B-Splines

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1 Introduction

General B-Splines provide a highly versatile approach to describe curves in computer graphics. B-Splines consist of sections of polynomial curves connected at points called knots. The polynomials of a given B-Spline all have the same degree, which is the degree of the B-Spline. The most used B-Splines consist of cubic segments, and are called “cubic B-Splines.” The following gives a definition of B-Splines, along with examples of some special cases.

2 Definitions

The $x$, $y$, and $z$ coordinates of the B-Spline curve is represented in parametric form as

$$x = x(t), \quad y = y(t), \quad z = z(t),$$  \hspace{1cm} (1)

where the parameter $t$ ranges over a prescribed set of values. (These three functions may be combined into a vector $\vec{X}(t)$ to simplify notation later.)

When a parametric curve is a B-Spline, then the parametric functions have some level of discontinuity at a set of parameter values called knots. For a B-Spline, there will be as many polynomial sections of the curve as the number of control points minus the degree of the polynomial. For $N$ control points and “n” as the degree of the polynomial there will be $N - n$ sections. These sections join at $N - n - 1$ knots. Let the control points be $\vec{p}_i$ for $i = 1, \ldots, N$. ($\vec{p}_i$ is a vector having components $x_i, y_i, z_i$.) If the dimensionality of the polynomial curve connecting the knots is $n$ then the parametric equation of the B-Spline is

$$\vec{X}(t) = \sum_{k=1}^{N} B^n_{k-1}(t)\vec{p}_k, \quad t_n \leq t \leq t_N, \quad N \geq n + 1.$$  \hspace{1cm} (2)

The knots must have values of the parameter $t$ associated with them. The values of $t$ at the knots are designated as $t_{j+n}$ for the knot joining the $j$th and the $j + 1$th polynomial segments. In addition there must be parameter values $t_n$ and $t_N$ corresponding to the beginning and ending of the complete curve. (There will be an addition $2n$ parameter values, $t_0, t_1, \ldots, t_{n-1}$ and $t_{N+1}, \ldots, t_{N+n}$, associated with the formulas for the blending polynomials given below, but these parameter values are never used in calculating the coordinates of the curve.) The values of the $t_j$ must be in monotonic increasing order as $j$ increases, but the values are not limited to being equally spaced, or to being integers, or to being positive.

The functions $B^n_k(t)$ are defined recursively as follows. First, the unit step function is defined as

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & \text{otherwise} \end{cases}.$$  \hspace{1cm} (3)

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Next, the zeroth order polynomial $B_0$ is defined as

$$B_0^0(t) = u(t - t_k)u(t_{k+1} - t).$$

(4)

Finally, the general order $B$ is given by

$$B_n^k(t) = \frac{t - t_k}{t_{k+n} - t_k} B_{n-1}^k(t) + \frac{t_{k+n+1} - t}{t_{k+n+1} - t_{k+1}} B_{n+1}^k(t).$$

(5)

Note that this definition means that $B_n^k$ is non-zero only in the range $t_k < t < t_{k+n+1}$.

### 3 Special properties of B-Splines

#### 3.1 Convex hull property of B-Splines

Note that by recursion one has:

$$B_n^k \geq 0$$

(6)

and that one can also show, one $n$ at a time, and ultimately by recursion,

$$\sum_{k=1}^{N} B_n^{(k-1)}(t) = 1, \quad t_n < t < t_N.$$  

(7)

Thus the convex hull requirements are met by B-Splines.

#### 3.2 Effect of repeated points

When adjacent control points have the same coordinates the B-Spline curve is brought closer to this coordinate. For $n$ identical adjacent control points the spline will interpolate the point.

### 4 Special case: integer knots, equally spaced

#### 4.1 General formulas

Suppose we let $t_k = k$. Then the definitions in eqs.(4) and (5) become

$$B_0^0(t) = u(t - k)u(k + 1 - t),$$

(8)

and

$$B_n^k(t) = \frac{t - k}{n} B_{n-1}^k(t) + \frac{k + n + 1 - t}{n} B_{n+1}^k(t).$$

(9)

Note that $B_0^0(t) = B_0^0(t - k)$. By recursion,

$$B_n^k(t) = B_0^0(t - k).$$

(10)

Thus for integer spacing of knots (hence uniform B-Splines) one only needs a single blending polynomial defined for each degree of spline.
4.2 Examples through cubic B-Splines

The polynomials of the higher degrees now become

\[ B_0^1(t) = tu(t)(1-t) + (2-t)u(t-1)u(2-t) \]  
\[ B_0^2(t) = \frac{1}{2}[t^2u(t)(1-t) + [t(2-t) + (3-t)(t-1)]u(t-1)u(2-t) + \]  
\[ (3-t)^2u(t-2)u(3-t)] \]  
\[ B_0^3(t) = \frac{1}{6}[t^3u(t)(1-t) + \]  
\[ [t^2(2-t) + t(t-1)(3-t) + (t-1)^2(4-t)]u(t-1)u(2-t) + \]  
\[ t(3-t)^2 + (t-1)(3-t)(4-t) + (t-2)(4-t)^2]u(t-2)u(3-t) + \]  
\[ (4-t)^3u(t-3)u(4-t)] \]  

Plots of the four lowest order blending polynomials are shown in Fig. 1.
4.3 Simplification for computer coding

A direct programming of eq.(2) for cubic B-Splines is not practical. Instead, an alternative is to program the drawing of each cubic section separately. Each such section will depend on four control points. Suppose, for example, there are only four control points. Then the single section has the parameteric equations

\[ X(t) = \frac{1}{6} [\bar{p}_1 (4-t)^3 + \bar{p}_2 ((t-1)(4-t)^2 + (t-2)(4-t)(5-t) + (t-3)(5-t)^2) + \bar{p}_3 ((t-2)^2(4-t) + (t-2)(t-3)(5-t) + (t-3)^2(6-t)) + \bar{p}_4 (t-3)^3], 3 \leq t \leq 4. \]  

Since the parameter \( t \) has a range from 3 to 4 in eq.(14), it makes sense to substitute a new parameter, \( s \), having a range of 0 to 1:

\[ s = t - 3 \quad t = s + 3 \]  

so that the parametric equation for the curve segment becomes

\[ \bar{X}(t) = \frac{1}{6} [\bar{p}_1 (1-s)^3 + \bar{p}_2 ((s+2)(1-s)^2 + (s+1)(1-s)(2-s) + s(2-s)^2) + \bar{p}_3 ((s+1)^2(1-s) + s(s+1)(2-s) + s^2(3-s)) + \bar{p}_4 s^3], 0 \leq s \leq 1. \]  

Now, one would draw this first curve segment, then change the control points in eq.(17) by increasing each \( \bar{p}_j \)'s subscript by one, and then draw the next segment, etc. The functions

\[ b_0(s) = \frac{(1-s)^3}{6} \]  
\[ b_1(s) = \frac{[(s+2)(1-s)^2 + (s+1)(1-s)(2-s) + s(2-s)^2]/6 \]  
\[ b_2(s) = \frac{[(s+1)^2(1-s) + s(s+1)(2-s) + s^2(3-s))]/6 \]  
\[ b_3(s) = \frac{s^3}{6} \]  

may be taken as “blending polynomials” for the cubic B-Spline sections. Then each section is formed as

\[ \bar{X}_j(s) = \sum_{i=0}^{3} b_i(s)\bar{p}_{i+j}, \quad 0 \leq s \leq 1, \quad j = 1, 2, \ldots, N-3. \]  

These polynomials are plotted in Fig. 2

If the same number of subdivisions of each section is desired, that is if \( s \) takes on the same set of values for each of the \( \bar{X}_j \) evaluations, then the values of the \( b_i(s) \) can be stored in arrays rather than being evaluated for each vertex call in the graphics routine.

References

Uniform cubic B-Spline segment blending polynomials

Fig. 2: Blending polynomials for cubic B-Spline curve segments.