Field of a Finite Line Charge

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The electric field intensity due to a constant line charge density, \( \rho_l \), along a straight line segment of length \( L \) can be used as a building block for constructing electric field intensity vectors for more complicated charge distributions. This electric field intensity will be derived in a coordinate independent manner in the following.

The geometry of the problem

![Diagram of a line charge with electric field intensity vectors](image)

Note the following definitions:

- The line segment has total length \( L \).
- The point where the electric field intensity \( \vec{E} \) is evaluated is located a perpendicular distance \( h \) from the line segment having constant line charge density \( \rho_l \).
- The angles between the perpendicular from the point of \( \vec{E} \) and lines drawn to the ends of the line segment of \( \rho_l \) are \( \alpha_1 \) and \( \alpha_2 \).
- A distance \( l \) is measured from the point of intersection of the perpendicular from the point of \( \vec{E} \) to the line of \( \rho_l \), positive when towards the end where \( \alpha_2 \) is the angle subtended.
- The angle \( \beta \) is measured from the perpendicular from the point of \( \vec{E} \) to the line charge, to the line from the point of \( \vec{E} \) to the location of \( l \) along the line charge. Thus the relationship, \( l = h \tan \beta \), holds.
- Unit vectors \( \hat{u}_\parallel \) and \( \hat{u}_\perp \) are defined as parallel to the line charge and perpendicular to it, respectively. These serve as basis vectors for expressing \( \vec{E} \).
Calculation of $\vec{E}$

The incremental charge, $dq$, contained in incremental length of the line, $dl$, is $dq = \rho l \, dl$. In turn, we have $dl = h \sec^2 \beta \, d\beta$. Thus the incremental contribution to $\vec{E}$ from $dq$ may be written as

$$d\vec{E} = \frac{1}{4\pi \varepsilon_0} \frac{\hat{u}_\perp h - \hat{u}_\parallel h \tan \beta}{[h \sec \beta]^3} \rho \, h \sec^2 \beta \, d\beta. \quad (1)$$

Simplifying eq.(1) and then integrating on $\beta$ gives the desired result for $\vec{E}$:

$$\vec{E} = \frac{\rho l}{4\pi \varepsilon_0 h} \int_{-\alpha_1}^{\alpha_2} (\hat{u}_\perp \cos \beta - \hat{u}_\parallel \sin \beta) \, d\beta$$

$$= \frac{\rho l}{4\pi \varepsilon_0 h} [\hat{u}_\perp (\sin \alpha_1 + \sin \alpha_2) + \hat{u}_\parallel (-\cos \alpha_1 + \cos \alpha_2)]. \quad (2)$$

Special cases

Infinite line charge

Suppose $L \gg h$. Then both $\alpha_1$ and $\alpha_2$ approach 90° so that

$$\vec{E} = \frac{\rho l}{2\pi \varepsilon_0 h} \hat{u}_\perp. \quad (3)$$

This gives a way to judge when a line segment is long enough to be considered infinitely long: when the $\alpha$ angles are close enough to right angles to make the cosine of the angle zero to the desired precision.

Half-infinite line charge

Another case of interest is when the point of observation of the field is at one end of an otherwise very long line charge. Suppose the end is the one to which $\alpha_2$ is measured. Then $\alpha_2 = 0$ and $\alpha_1 = 90^\circ$. The result for $\vec{E}$ is

$$\vec{E} = \frac{\rho l}{4\pi \varepsilon_0 h} (\hat{u}_\perp + \hat{u}_\parallel). \quad (4)$$