Lagrange Multipliers for Quadratic Forms With Linear Constraints

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When one requires an extremum of a quadratic form

$$W = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij} V_i V_j \tag{1}$$

subject to the linear constraints

$$L_m = \sum_{l=1}^{N} B_{ml} V_l - D_m = 0, \qquad m = 1, 2, ...P \tag{2}$$

then the method of Lagrange multipliers[1] may be applied as follows.
Let $W_2 = W + \sum_{m=1}^{P} \lambda_m L_m$:

$$W_2 = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij} V_i V_j + \sum_{m-1}^{P}\sum_{l=1}^{N} \lambda_m B_{ml} V_l - \sum_{m=1}^{P} \lambda_m D_m. \tag{3}$$

Now, to find an extreme of W, set the partial derivatives of W_2 with respect to the V_l's to zero, and combine with eq.(2) to obtain a set of $N + P$ linear equations to solve for the N values of the V_i and the P values of the Lagrange multipliers λ_m.

$$\frac{\partial W_2}{\partial V_k} = \frac{1}{2}\sum_{i=1}^{N} A_{ik} V_k + \frac{1}{2}\sum_{j=1}^{N} A_{kj} V_k + \sum_{m=1}^{P} \lambda_m B_{mk} = 0. \tag{4}$$

Since eqs.(2) and (4) are linear, they may be written in matrix form:

$$A_{N\times N} = \{A_{ij}\}, \qquad B_{P\times N} = \{B_{ml}\}, \qquad V_{N\times 1} = \{V_i\}, \qquad D_{P\times 1} = \{D_m\}, \qquad \lambda_{P\times 1} = \{\lambda_m\}. \tag{5}$$

Let

$$C_{N\times N} = \frac{1}{2}(A + A^T). \tag{6}$$

Then eq.(4) becomes

$$CV + B^T \lambda = 0, \tag{7}$$

and eq.(2) becomes

$$BV = D. \tag{8}$$

Now eqs.(7) and (8) may be placed in a partitioned matrix equation as

$$\begin{bmatrix} C & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} V \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ D \end{bmatrix}. \tag{9}$$

Eq.(9) has the symbolic solution

$$\begin{bmatrix} V \\ \lambda \end{bmatrix} = \begin{bmatrix} C & B^T \\ B & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ D \end{bmatrix} \tag{10}$$

References

[1] I. S. Sokolnikoff and R. M. Redheffer, *Mathematics of Physics and Modern Engineering*, 2nd.ed.,McGraw-Hill, 1966, pp. 345-347.