Electromagnetic potentials for time varying fields

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1 Fundamental equations

The fundamental relationships for time varying electromagnetic fields are Maxwell's equations, the Lorentz force law, and the auxiliary equations associated with material properties.

1.1 Maxwell's equations

$$\nabla \cdot \overline{D} = \rho_v \tag{1}$$
$$\nabla \cdot \overline{B} = 0 \tag{2}$$
$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \tag{3}$$
$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t} \tag{4}$$

1.2 Material relationships

$$\overline{D} = \epsilon_0 \overline{E} + \overline{P} \tag{5}$$
$$\overline{B} = \mu_0 \overline{H} + \mu_0 \overline{M} \tag{6}$$

1.2.1 For isotropic materials:

$$\overline{D} = \epsilon\overline{E} \tag{7}$$
$$\overline{B} = \mu\overline{H} \tag{8}$$
$$\overline{J} = \sigma\overline{E} \tag{9}$$
$$\tag{10}$$

1.3 Lorentz force

$$\overline{F} = q(\overline{E} + \overline{u} \times \overline{B}) \tag{11}$$

2 Potentials

In electrostatics, the electric potential, V, was defined such that $\overline{E} = -\nabla V$. But the vector identity

$$\nabla \times \nabla V \equiv 0 \tag{12}$$

along with the Maxwell's equation for $\nabla \times \overline{E}$ shows that $\overline{E} = -\nabla V$ is not valid where a time varying magnetic field is present. Thus a new approach to potentials is required for time varying fields.

2.1 The Helmholtz theorem[1]

Any vector field, $\overline{F}$, which has divergence $\nabla \cdot \overline{F} = s$ and curl $\nabla \times \overline{F} = \overline{c}$ such that both s and $\overline{c}$ vanish at infinite distances from the region of interest, may be expressed as the sum of two fields, one of which has a zero curl and the other of which has a zero divergence.

2.2 Magnetic vector potential

As a consequence of the Helmholtz theorem and the Maxwell's equations involving $\overline{B}$ and $\overline{H}$, one sees that $\overline{B}$ has only the part with zero divergence. But the vector identity

$$\nabla \cdot \nabla \times \overline{F} \equiv 0 \tag{13}$$

[1] Proof of this theorem is in: George Arfken, *Mathematical Methods for Physicists*, 2nd. ed., Academic Press, 1970, pp.66-70.

means that one may express $\overline{B}$ as the curl of another field, $\overline{A}$. This $\overline{A}$ is called the *vector magnetic potential.*

$$\overline{B} = \nabla \times \overline{A}. \tag{14}$$

2.3 Potentials for the time varying electric field

Placing eq.(14) in eq.(3) gives

$$\nabla \times \overline{E} = -\nabla \times (\partial \overline{A}/\partial t). \tag{15}$$

With eq.(15) giving the curl of the electric field, one may now extend the concept of the electric potential to a *scalar* potential, V, which combined with the *vector* potential gives the electric field:

$$\overline{E} = -\nabla V - \partial \overline{A}/\partial t. \tag{16}$$

2.4 Maxwell's equations in terms of potentials

With the electric field given by eq.(16) and the magnetic flux density given by eq.(15), the Maxwell equations without sources, eqs.(2) and (3), are automatically satisfied. One may then use the potentials, along with the material relationships, in the remaining Maxwell equations to have just two equations in terms of the potentials:

$$\nabla^2 V + \frac{\partial(\nabla \cdot \overline{A})}{\partial t} = \frac{1}{\epsilon_0}(\nabla \cdot \overline{P} - \rho_v) \tag{17}$$

$$\nabla \times (\nabla \times \overline{A}) + \mu_0 \epsilon_0 \frac{\partial^2 \overline{A}}{\partial t^2} + \mu_0 \epsilon_0 \frac{\partial \nabla V}{\partial t} = \mu_0(\overline{J} + \nabla \times \overline{M}) \tag{18}$$

2.5 Gauge condition

We note that only the curl of $\overline{A}$ is specified by its definition in terms of $\overline{B}$. According to the Helmholtz theorem, both the divergence and curl of $\overline{A}$ must be given to enable determining $\overline{A}$. Thus eqs.(17) and (18) must be supplemented by a specification of $\nabla \cdot \overline{A}$. This specification of the divergence of $\overline{A}$ is called the choice of *gauge.*

There are two special choices of gauge that have names given to them:

1. Coulomb gauge: $\nabla \cdot \overline{A} = 0$

   and

2. Lorentz gauge: $\nabla \cdot \overline{A} = -\mu_0 \epsilon_0 \partial V / \partial t$.

The Coulomb gauge is so named because with this choice the equation for the scalar potential, eq.(17), becomes the same as for the electrostatic case.

The Lorentz gauge decouples the two potential equations, eqs.(17) and (18). This gives two wave equations to be satisfied by V and $\overline{A}$. [2]

2.5.1 Vector identity for curl of curl

In order to simplify eq.(18) one needs the identity

$$\nabla \times (\nabla \times \overline{A}) \equiv \nabla(\nabla \cdot \overline{A}) - \nabla^2 \overline{A}. \tag{19}$$

2.5.2 Equations for potentials in Coulomb gauge

When $\nabla \cdot \overline{A} = 0$

$$\nabla^2 V = -(\rho_v - \nabla \cdot P)/\epsilon_0 \tag{20}$$

$$\nabla^2 \overline{A} - \mu_0 \epsilon_0 \frac{\partial^2 \overline{A}}{\partial t^2} = \mu_0 \epsilon_0 \frac{\partial \nabla V}{\partial t} - \mu_0 (\overline{J} + \nabla \times \overline{M}) \tag{21}$$

2.5.3 Equations for potentials in Lorentz gauge

When $\nabla \cdot \overline{A} = -\mu_0 \epsilon_0 \partial V / \partial t$

$$\nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0}(\rho_v - \nabla \cdot \overline{P}) \tag{22}$$

$$\nabla^2 \overline{A} - \mu_0 \epsilon_0 \frac{\partial^2 \overline{A}}{\partial t^2} = -\mu_0 (\overline{J} + \nabla \times \overline{M}) \tag{23}$$

2.5.4 Equations for potentials in homogeneous, isotropic materials

In a material in which μ and ϵ are constants, in eqs.(20) through (23), replace μ_0 by μ, replace ϵ_0 by ϵ, and set $\overline{M} = \overline{P} = 0$.

[2] This makes the four vector $(c\overline{A}, V)$ manifestly Lorentz covariant. See Panofsky and Phillips, *Classical Electricity and Magnetism*,2nd.ed.,Addison-Wesley, 1962, chapter 18, for an explanation of what this means.)