Transmission line equations in phasor form

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The text for this class presents transmission lines in phasor form by first doing all the lossless line cases, then adding in the losses in a later section. The text approach is not wrong, and is adequate for understanding what is needed for the problems. Other texts use a different approach. In the following, the general case is presented first and then specialized to the lossless one. Also, the space coordinate used is taken as distance from the end of the line rather than distance from the beginning. This is more in keeping with the approach needed for use with Smith charts.

1 The basic equations for transmission lines in phasor form

A transmission line has distributed impedance parameters. Let the series impedance per unit length be $Z$ and the parallel admittance per unit length be $Y$. Then the line shown in the circuit on the left in Fig. 1 has an infinitesimal section as shown on the right. Note that the coordinate used is $l$, which is distance from the end of the line. Using Kirchhoff’s laws on the infinitesimal section of the line gives the differential equations relating current and voltage:

$$\frac{dV}{dl} = ZI \quad (1)$$

$$\frac{dI}{dl} = YV. \quad (2)$$
Taking the derivative of eq.(1) and substituting eq.(2) into it gives the second order differential equation for the voltage as a function of distance from the end of the line:

$$\frac{d^2 V}{dl^2} = Z\gamma V.$$  \hfill (3)

Eq.(3) has the general solution

$$V(l) = V^+ e^{\gamma l} + V^- e^{-\gamma l} = V^+ e^{\gamma l}(1 + \Gamma_0 e^{-2\gamma l})$$  \hfill (4)

where

$$\gamma = \alpha + j\beta = \sqrt{Z\gamma}$$  \hfill (5)

with $\alpha$ and $\beta$ real and $\beta \geq 0$. $V^+$ is the amplitude of the voltage wave traveling towards the end of the line, and $V^-$ is the amplitude of the voltage wave traveling towards the beginning of the line. Also, $\Gamma_0 = V^-/V^+$ is the reflection coefficient at $l = 0$.

Next, using eq.(2) on eq.(4) yields the equation for current on the line:

$$I(l) = \frac{V^+}{Z_0} e^{\gamma l}(1 - \Gamma_0 e^{-2\gamma l})$$  \hfill (6)

where

$$Z_0 = \sqrt{\frac{Z}{\gamma}}$$  \hfill (7)

is the characteristic impedance of the line.

Taking the ratio of the voltage to the current at any location along the line yields the impedance value at that location:

$$Z(l) = \frac{V(l)}{I(l)} = Z_0 \frac{1 + \Gamma_0 e^{-2\gamma l}}{1 - \Gamma_0 e^{-2\gamma l}}.$$  \hfill (8)
To obtain a simpler equation, let \( z = \frac{Z(l)}{Z_0} \) and let \( \Gamma = \Gamma_0 e^{-2\gamma l} \) so that

\[
z = \frac{1 + \Gamma}{1 - \Gamma}.
\]  

(9)

Eq.(9) may be solved for \( \Gamma \):

\[
\Gamma = \frac{z - 1}{z + 1}.
\]  

(10)

Now, at the end of the line, the ratio of voltage to current must be the load impedance \( Z_L \). Thus

\[
\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0}.
\]  

(11)

At the input to the line the input impedance will then be the ratio of voltage to current at the input, \( Z_{in} \). We thus have from eq.(9)

\[
Z_{in} = Z_0 \frac{1 + \Gamma_{in}}{1 - \Gamma_{in}}
\]  

(12)

where

\[
\Gamma_{in} = \Gamma_0 e^{-2\gamma L}.
\]  

(13)

With the input impedance to the transmission line available, one may obtain the values of \( V(L) \) and \( I(L) \) from the generator voltage and impedance, as shown in the circuit on the left in Fig. 1.

\[
V(L) = V_G \frac{Z_{in}}{Z_G + Z_{in}}
\]  

(14)

\[
I(L) = V_G \frac{1}{Z_G + Z_{in}}.
\]  

(15)

Then eq.(4) may be used to find the incident voltage amplitude:

\[
V^+ = \frac{V(L)e^{-\gamma L}}{1 + \Gamma_{in}}.
\]  

(16)

With \( V^+ \) known, the voltage and current at any point on the line may be found.
2 Applications

2.1 Using wavelength units for distance

When the propagation constant \( \gamma = \alpha + j\beta \) has been found, so that \( \beta \) is known for a transmission line, it is convenient to express distances along the line in terms of the wavelength. Since the wavelength is \( \lambda = 2\pi/\beta \), in the equations where \( e^{\gamma l} \) is a factor, this factor may be written as

\[
e^{\gamma l} = e^{\alpha l} e^{j\beta l} = e^{2\pi \frac{\alpha}{\beta} l} e^{2\pi \frac{l}{\lambda}}
\] (17)

Eq.(17) shows that the \( \beta \) factor adds a phase shift of \( 2\pi \) radians (360°) for each wavelength of distance.

For the reflection coefficient,

\[
\Gamma(l) = \Gamma_0 e^{-2\gamma l} = \Gamma_0 e^{-4\pi \frac{\alpha}{\beta} l} e^{-4\pi \frac{l}{\lambda}}
\] (18)

one sees that a distance of one wavelengths translates into a phase shift of \(-720°\). Thus it is easy to find \( \Gamma \) at the input to the given the \( \Gamma \) at the load and the length of the line in wavelengths, especially for a lossless line where \( \alpha = 0 \).

2.2 Lossless lines

For lossless lines, the per-unit-length impedance and admittance become \( Z = j\omega L \) and \( Y = j\omega C \). Thus

\[
\gamma = \alpha + j\beta = \sqrt{j\omega L j\omega C} = j\omega \sqrt{LC}
\] (19)

or \( \alpha = 0 \) and \( \beta = \omega \sqrt{LC} \), which leads to a speed of propagation on the line of \( u = 1/\sqrt{LC} \).

In the same way the characteristic impedance becomes

\[
Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{L}{C}}.
\] (20)

2.2.1 Input impedance to a lossless line

Given that a line is lossless, one finds the input impedance to the line by calculating the reflection coefficient at the load as \( \Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} \) and then
applying the phase shift of $-720^\circ$ times the length of the line in wavelengths to get $\Gamma_{in}$. Then $Z_{in} = Z_0 \frac{1+\Gamma_{in}}{1-\Gamma_{in}}$.

If the load on the line is a short circuit, then the result is

$$Z_{in-sc-load} = Z_0 \frac{1 - e^{-4\pi l/\lambda}}{1 + e^{-4\pi l/\lambda}} = Z_0 \frac{e^{2\pi l/\lambda} - e^{-2\pi l/\lambda}}{e^{2\pi l/\lambda} + e^{-2\pi l/\lambda}} = jZ_0 \tan(2\pi l/\lambda). \quad (21)$$

Thus a section of short circuited, lossless transmission line has an input impedance that is a pure reactance. The magnitude can have any value depending on the length of the line.